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Optimal Gaussian Solutions of Nonlinear Stochastic Partial Differential Equations

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We present a linearization procedure of a stochastic partial differential equation for a vector field $(X_i(t, x))$ $(t \in [0, \infty),\ x \in R^d,\ i = 1,..., n)$: $\partial_t X_i(t, x) = b_i(X(t, x)) + D_i \Delta X_i(t, x) + \sigma_i f_i(t, x)$. Here Δ is the Laplace–Beltrami operator in R^d, and $(f_i(t, x))$ is a Gaussian random field with $\langle f_i(t, x) f_j(t', x')\rangle = \delta_{ij}\,\delta(t - t')\,\delta(x - x')$. The procedure is a natural extension of the equivalent linearization for stochastic ordinary differential equations. The linearized solution is optimal in the sense that the distance between true and approximate solutions is minimal when it is measured by the Kullback–Leibler entropy. The procedure is applied to the scalar-valued Ginzburg–Landau model in R^1 with $b_1(z) = \mu z - \nu z^3$. Stationary values of mean, variance, and correlation length are calculated. They almost agree with exact ones if $\mu \lesssim 1.24\ (\nu^2\sigma_1^4/D_1)^{1/3} := \mu_c$. When $\mu \geqslant \mu_c$, there appear quasistationary states fluctuating around one of the bottoms of the potential $U(z) = \int b_1(z)\,dz$. The second moment at the quasistationary states almost agrees with the exact one. Transient phenomena are also discussed. Half-width at half-maximum of a structure function decays like $t^{-1/2}$ for small t. The diffusion term $\partial_x^2 X$ accelerates the relaxation from the neighborhood of an unstable initial state $X(0, x) \simeq 0$.

KEY WORDS: Equivalent linearization; nonlinear stochastic partial differential equation; Kullback–Leibler entropy; Ginzburg–Landau model; relaxation from unstable state.

1. INTRODUCTION

The equivalent linearization[1,2] (EL), which some authors recently call piecewise optimal linearization[3] or approximate Gaussian representation of evolution equation,[4] seeks approximate Gaussian solutions of nonlinear

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diffusion processes. Suppose that we are given a stochastic ordinary differential equation for a scalar $X = (X(t))$ with a constant diffusion σ:

$$dX(t)/dt = b(X(t)) + \sigma f(t) \tag{1.1}$$

where $f(t)$ is a Gaussian white noise with $\langle f(t) \rangle = 0$ and $\langle f(t) f(t') \rangle = \delta(t - t')$. Here the symbol $\langle \cdot \rangle$ indicates the ensemble average. An approximate Gaussian solution $Y = (Y(t))$ satisfying

$$dY(t)/dt = \alpha(t) + \beta(t)[Y(t) - m(t)] + \sigma f(t) \tag{1.2}$$

is obtained in a self-consistent way by making $\alpha(t)$, $\beta(t)$ minimize

$$\langle \{b(Y(t)) - \alpha(t) - \beta(t)[Y(t) - m(t)]\}^2 \rangle \tag{1.3}$$

Theoretical aspects of the EL have been studied; Nakazawa[2] showed that the EL is the first order approximation of the Wiener–Hermite expansion of the original process X. Murakami[5] and Ito[6] proved that among all Gaussian processes Y satisfying (1.2), the one minimizing (1.3) is the closest to the original process X when the distance between X and Y is measured by the Kullback–Leibler entropy. Numerical calculation by Valsakumar *et al.*[3] and West *et al.*[4] showed that the EL is satisfactory from a practical point of view.

This paper aims at presenting a counterpart of the EL for a stochastic partial differential equation (SPDE) for a vector field $X = (X_i(t, x))$ of the form

$$\partial_t X_i(t, x) = b_i(X(t, x)) + D_i \Delta X_i(t, x) + \sigma_i f_i(t, x), \qquad i = 1, 2, \dots, n,\ x \in R^d \tag{1.4}$$

Here $b(z) = (b_i(z))$ is a vector function, Δ is the Laplace–Beltrami operator in R^d, $f(t, x) = (f_i(t, x))$ is a Gaussian white random field with

$$\begin{aligned} &\langle f_i(t, x) \rangle = 0, \\ &\langle f_i(t, x) f_j(t', x') \rangle = \delta_{ij} \delta(t - t')\, \delta(x - x') \end{aligned} \tag{1.5}$$

and $D_i > 0$, $\sigma_i \neq 0$ are constants.

Equation (1.4) describes spatially extended systems under the effects of diffusion and random forces which are represented by the second and the third terms on the right hand side of (1.4). Applications of (1.4) are found in Ref. 15 to hydrodynamics, nonlinear optics, chemical reaction models, and other related problems.

In Section 2 the EL formula is derived for (1.4) with $d = n = 1$, and the assertion by Murakami and Ito is shown to be valid. General case (1.4) is discussed in Section 3.

Section 4 is devoted to application of the EL to the scalar-valued Ginzburg–Landau model in R^1, i.e., (1.4) with $n=d=1$, $b_1(z)=\mu z-\nu z^3$. In Section 4.1, SPDE (1.4) is scaled such that $\nu=D_1=\sigma_1=1$, and basic equations for mean and variance and derived. In Section 4.2, stationary solution is discussed. We find the following:

(1) Spatial correlation function $V(x)$ takes a correct form $\exp(-\text{const}\,|x|)$ (const >0); EL improves a defect of the approximation by Langer[7] and Tomita and Murakami.[8,9]

(2) There exists a critical value $\mu_c \simeq 1.24$ such that we have a unique stationary solution if $\mu<\mu_c$ and five stationary solutions if $\mu>\mu_c$; one is a true stationary state, two of them represent quasistationary states fluctuating around $\pm\mu^{1/2}$, bottoms of the potential $U(z)=\int_0^z b_1(z)\,dz$, and the other two are spurious because they are locally unstable.

(3) When $\mu<\mu_c$, variance and correlation length almost agree with exact ones obtained numerically by Scalapino *et al.*[10]

(4) Even when $\mu>\mu_c$, the quasistationary states give almost correct second order moment.

The time-dependent problem is treated in Section 4.3. Relaxation from an unstable initial state $X(0,x)\simeq 0$ is studied numerically; the diffusion term $\partial_x^2 X$ accelerates the relaxation considerably. The $t^{-1/2}$ behavior of half-width at half-maximum of a structure function[9] for small time t is confirmed also in the EL.

The EL of the present paper approximates the original field X by a Gaussian field with spatial uniformity in the sense of ensemble average. In Section 5, a discussion is given on this point and a possible improvement is suggested.

Stability of multiple stationary states is discussed in the Appendix.

2. EQUIVALENT LINEARIZATION OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

Consider a scalar stochastic field $X=(X(t,x))$ determined by a stochastic partial differential equation (SPDE) (1.4) with $n=d=1$:

$$\partial_t X(t,x)=b(X(t,x))+D\,\partial_x^2 X(t,x)+\sigma f(t,x), \qquad x\in R^1 \tag{2.1}$$

Here $b(z)$ is a scalar function, and $D>0$, $\sigma\neq 0$ are constants. Gaussian random field $f(t,x)$ satisfies (1.5) with $n=d=1$. Equation (2.1) is to be understood as

$$dX(t,x)=b(X(t,x))\,dt+D\,\partial_x^2 X(t,x)\,dt+\sigma\,dW(t,x) \tag{2.2}$$

Here $W(t, x)$ is the so-called cylindrical Brownian motion which formally satisfies $f(t, x) = \partial_t W(t, x)$. See Funaki[11] for further mathematical discussion.

We suppose that the system determined by SPDE (2.1) is spatially uniform in the sense of ensemble average. Let us find an approximate Gaussian field $Y = (Y(t, x))$ given by

$$\partial_t Y(t, x) = \{\alpha(t) + \beta(t)[Y(t, x) - m(t)]\} + D\,\partial_x^2 Y(t, x) + \sigma f(t, x) \tag{2.3}$$

with

$$m(t) = \langle Y(t, x)\rangle \tag{2.4}$$

Nonrandom functions $\alpha(t)$, $\beta(t)$ are obtained by minimizing

$$I(t) := \langle\{b(Y(t, x)) - \alpha(t) - \beta(t)\,\xi(t, x)\}^2\rangle \tag{2.5}$$

where

$$\xi(t, x) := Y(t, x) - m(t) \tag{2.6}$$

$I(t)$ attains its minimum when

$$\alpha(t) = \langle b(Y(t, x))\rangle := \bar{\alpha}(t) \tag{2.7}$$

$$\beta(t) = \langle b(Y(t, x))\,\xi(t, x)\rangle / \langle\xi(t, x)^2\rangle := \bar{\beta}(t) \tag{2.8}$$

since

$$I_t = (\alpha_t - \bar{\alpha}_t)^2 + \langle\xi_{t,x}^2\rangle(\beta_t - \bar{\beta}_t)^2 - \langle b(Y_{t,x})\,\xi_{t,x}\rangle^2/\langle\xi_{t,x}^2\rangle + \langle b(Y_{t,x})^2\rangle - \langle b(Y_{t,x})\rangle^2$$

The EL scheme consists of solving (2.3), (2.4), (2.7), and (2.8) self-consistently.

Let us derive a closed set of equations for nonrandom quantities. By taking the average of both sides of (2.3) we have

$$\dot{m}(t) = \alpha(t) \tag{2.9}$$

ξ satisfies an SPDE

$$\partial_t \xi(t, x) = \beta(t)\,\xi(t, x) + D\,\partial_x^2 \xi(t, x) + \sigma f(t, x) \tag{2.10}$$

It is solved by using the Fourier transformation as

$$\xi(t, x) = (2\pi)^{-1} H(t) \int dk\, e^{-ikx} e^{-Dk^2 t} \int dy\, e^{iky} \xi(0, y) + \sigma (2\pi)^{-1} H(t) \int dk\, e^{-ikx} \int_0^t ds\, H(s)^{-1} e^{Dk^2(s-t)} \tilde{f}(s, k) \tag{2.11}$$

Here

$$H(t) = \exp\left[\int_0^t \beta(s)\, ds\right] \tag{2.12}$$

and $\tilde{f}$ is the Fourier transform of f:

$$\tilde{f}(t, k) = \int dx\, e^{ikx} f(t, x) \tag{2.13}$$

satisfying

$$\begin{aligned} \langle \tilde{f}(t, k) \rangle &= 0 \\ \langle \tilde{f}(t, k)\, \tilde{f}(t', k') \rangle &= 2\pi\delta(t - t')\, \delta(k + k') \end{aligned} \tag{2.14}$$

Define spatial correlation function V as

$$V(t, x - x') := \langle \xi(t, x)\, \xi(t, x') \rangle \tag{2.15}$$

which depends only on $x - x'$ by spatial uniformity assumption. Substituting (2.11) into (2.15) and using (2.14), we have

$$\begin{aligned} V(t, x) &= (2\pi)^{-1} H(t)^2 \int dk\, e^{-ikx} e^{-2Dk^2 t} \int dy\, e^{iky} V(0, y) \\ &\quad + \sigma^2 (2\pi)^{-1} H(t)^2 \int dk\, e^{-ikx} \int_0^t ds\, H(s)^{-2}\, e^{2Dk^2(s-t)} \end{aligned} \tag{2.16}$$

or

$$\partial_t V(t, x) = 2\beta(t)\, V(t, x) + 2D\, \partial_x^2 V(t, x) + \sigma^2 \delta(x) \tag{2.17}$$

Here we have assumed that the initial value $\xi(0, x)$ is spatially uniform, i.e., $\langle \xi(0, x)\, \xi(0, x') \rangle$ depends only on $x - x'$; and that $\xi(0, x)$ has no correlation with $f(t, x)$ i.e., $\langle f(t, x)\, \xi(0, x') \rangle \equiv 0$. We note that α and β given by (2.7), (2.8) are expressed by $V(t, 0)$ since ξ is a Gaussian with mean zero; therefore (2.9), (2.17) with (2.7), (2.8) form a closed set of equations for $m(t)$, $V(t, x)$. In Section 4, we carry out a further analysis for Ginzburg–Landau model given by $b(z) = \mu z - \nu z^3$.

Let us see how the EL is optimal in the sense of entropy. Consider SPDE (2.1) or (2.2) on a finite space interval $(-L, L)$. For each t, $X(t, \cdot)$ takes values in $\mathfrak{C}$, the space of continuous functions defined on $(-L, L)$. Hence the SPDE (2.1) determines a probability measure P^X in $\mathscr{C}$, the space of $\mathfrak{C}$-valued continuous functions defined on $[0, \infty)$, through a relation: for any Borel subset A of $\mathscr{C}$, $P^X(A) = P(\omega \in \Omega; X(\cdot, \cdot, \omega) \in A)$. Here by

$(\Omega, \mathscr{F}, P)$ we denote the basic probability space on which the SPDE (2.1) is defined. Let P^Y denote a probability measure induced by Y given by (2.3). Let us consider the Kullback–Leibler entropy defined by

$$S^L(X, Y) = -\int_{\mathscr{C}} dP^Y(y) \log dP^X/dP^Y(y) \tag{2.18}$$

where dP^X/dP^Y is the Radon–Nikodym derivative. As known,[12] $S^L(X, Y) \geqslant 0$, and $=0$ if and only if $P^X = P^Y$. S^L can be used as a quantity representing the difference between true process X and approximate one Y. To calculate S^L, we first divide spatial interval $(-L, L)$ into $2N$ segments, and consider $(2N+1)$-dimensional stochastic ordinary differential equations for $X_N = (X(t, kL/N))$ $(k = -N,..., N)$:

$$\begin{aligned} dX_N(t, kL/N) = b(X_N(t, kL/N))\, dt + D\, \Delta_N X_N(t, kL/N)\, dt \\ + (N/L)^{1/2} \sigma\, dW(t, k) \end{aligned} \tag{2.19}$$

Here

$$\begin{aligned} \Delta_N X_N(t, kL/N) = N^2 L^{-2}[X_N(t, (k+1)L/N) - 2X_N(t, kL/N) \\ + X_N(t, (k-1)L/N)] \end{aligned} \tag{2.20}$$

and $W(\cdot, k)$ $(k = -N,..., N)$ is a $(2N+1)$-dimensional Wiener process. We associate subscript N to represent quantities corresponding to the processes X_N and Y_N; $P_N^X(A) := P(\omega; X_N \in A)$, etc. The Radon–Nikodym derivative dP_N^X/dP_N^Y is given by the Girsanov formula[13]:

$$\begin{aligned} dP_N^X/dP_N^Y(y) = \exp \Bigg[\int_0^t (L/N)^{1/2} \sum_{k=-N}^{N} \phi(t, kL/N)\, dW(t, k) \\ - \tfrac{1}{2} \int_0^t (L/N) \sum_{k=-N}^{N} \phi^2(t, kL/N)\, dt\Bigg] \Bigg|_{Y=y} \end{aligned} \tag{2.21}$$

Here

$$\begin{aligned} \phi(t, kL/N) = \sigma^{-1}\{b(Y(t, kL/N)) - \alpha(t) \\ - \beta(t)[Y(t, kL/N) - m(t)]\} \end{aligned} \tag{2.22}$$

As shown by Funaki,[11] $P_N^X \to P^X$ and $P_N^Y \to P^Y$ as $N \to \infty$. Therefore

$$\begin{aligned} S^L(X, Y) &= \lim_{N\to\infty} S_N^L(X, Y) = \lim_{N\to\infty} \tfrac{1}{2} L \int_0^t \sum_{k=-N}^{N} N^{-1} \langle \phi^2(t, kL/N) \rangle\, dt \\ &= \tfrac{1}{2} L \int_0^t \int_{-L}^{L} \langle \phi^2(t, x) \rangle\, dt\, dx \end{aligned} \tag{2.23}$$

When L is large, the Kullback–Leibler entropy per unit length $\lim_{L\to\infty} S^L(X, Y)/(2L)$ becomes

$$\frac{1}{4}\int_0^t \int_{-\infty}^{\infty} \langle \phi^2(t, x)\rangle \, dt \, dx \tag{2.24}$$

Since we have assumed the spatial uniformity, we observe that minimization of (2.24) is equivalent to that of $I(t)$ defined by (2.5).

We finally remark that as far as we assume the spatial uniformity, we do not have to take into account a term $\partial_x[Y(t,x) - m(t)]$ in (2.3). In other words, if X is approximated by a Gaussian field Y following

$$\begin{aligned} \partial_t Y(t, x) = {} & \{\alpha(t) + \beta(t)[Y(t, x) - m(t)]\} \\ & + \gamma(t)\, \partial_x Y(t, x) + D\, \partial_x^2 Y(t, x) + \sigma f(t, x) \end{aligned} \tag{2.25}$$

then optimal choice is $\gamma = 0$. In fact, α, β, γ which minimize $I(t)$ defined by

$$I(t) := \langle [b(Y(t, x)) - \alpha(t) - \beta(t)\, \xi(t, x) - \gamma(t)\, \partial_x \xi(t, x)]^2 \rangle$$

involve a term of the form $\langle \xi(t, x)\, \partial_x \xi(t, x)\rangle = \partial_x V(t, 0)$. Here ξ and V are defined by (2.6) and (2.15), respectively. $V(t, x)$ follows a parabolic equation like (2.17), which forces $\partial_x V(t, 0)$ to depend on sign x, a term violating spatial uniformity assumption.

3. EL IN GENERAL CASE

We extend the results in the previous section to a general case (1.4). When $d \geqslant 2$, (1.4) with some physically suitable conditions for $|x| \to \infty$ will not be well posed; Consider, for example, (1.4) with $n = 1$, $d \geqslant 2$, $b(z) = -\mu z$ ($\mu > 0$). Then $V(t, x) := \langle X(t, x)\, X(t, 0)\rangle$ satisfies a linear parabolic equation

$$\partial_t V(t, x) = -2\mu V(t, x) + 2D\, \Delta V(t, x) + \sigma^2 \delta(x) \tag{3.1}$$

which involves infinity: $V(t, 0) = \infty$. To avoid the difficulty we introduce a suitable boundary condition on a finite domain. Here we impose a periodic boundary condition:

$$\begin{aligned} X(t, x) &= X(t, x + 2L) \\ \partial_x X(t, x) &= \partial_x X(t, x + 2L), \qquad \forall x \in R^d \end{aligned} \tag{3.2}$$

with some constant vector $L \in R^d$. Approximate Gaussian field $Y(t, x)$ with the boundary condition (3.2) is assumed to have a form

$$\begin{aligned} \partial_t Y_i(t, x) = {} & \left\{ \alpha_i(t) + \sum_{j=1}^{n} \beta_{ij}(t)[Y_j(t, x) - m_j(t)] \right\} \\ & + D_i\, \Delta Y_i(t, x) + \sigma_i f_i(t, x) \end{aligned} \tag{3.3}$$

with

$$m_i(t) = \langle Y_i(t, x) \rangle \tag{3.4}$$

$\alpha(t) = (\alpha_i(t))$ and $\beta(t) = (\beta_{ij}(t))$ are determined by minimizing the Kullback–Leibler entropy, or equivalently by minimizing

$$I(t) := \left\langle \sum_{i=1}^{n} \sigma_i^{-2} [b_i(Y(t, x)) - \alpha_i(t) - \sum_{j=1}^{n} \beta_{ij}(t)\, \xi_j(t, x)]^2 \right\rangle$$

where

$$\xi_i(t, x) := Y_i(t, x) - m_i(t) \tag{3.5}$$

Define $n \times n$ matrices θ and V by

$$\theta(t) = (\langle b_i(Y(t, x))\, \xi_j(t, x) \rangle) \tag{3.6}$$

$$V(t, x) = (\langle \xi_i(t, x)\, \xi_j(t, 0) \rangle) \tag{3.7}$$

Associate the symbol $\sim$ when we multiply a matrix σ^{-1} from the left; $\tilde{\alpha} = \sigma^{-1}\alpha$, etc., where $\sigma = \mathrm{diag}(\sigma_i)$. After simple calculation $I(t)$ is written as

$$\begin{aligned} I(t) = {} & |\tilde{\alpha}_t - \langle \tilde{b}_{t,x} \rangle|^2 + \mathrm{tr}[V_{t,0}(\tilde{\beta}_t - \tilde{\theta}_t V_{t,0}^{-1})^* (\tilde{\beta}_t - \tilde{\theta}_t V_{t,0}^{-1})] \\ & + \langle |\tilde{b}_{t,x}|^2 \rangle - |\langle \tilde{b}_{t,x} \rangle|^2 - \mathrm{tr}[V_{t,0}^{-1} \tilde{\theta}_t^* \tilde{\theta}_t] \end{aligned} \tag{3.8}$$

Here $|\cdot|$ denotes the Euclidean norm in R^n, A^* is a transpose of A and $b(t, x) = (b_i(Y(t, x)))$. Since $V_{t,0}$ is symmetric and positive definite, the second term on the right-hand side of (3.8) is nonnegative; hence α and β which minimize $I(t)$ are given by

$$\alpha(t) = \langle b(Y(t, x)) \rangle \tag{3.9}$$

$$\beta(t) = \theta(t)\, V(t, 0)^{-1} \tag{3.10}$$

$m(t)$ and $V(t, x)$ follow matrix equations

$$\begin{aligned} \dot{m}(t) &= \alpha(t) \\ \partial_t V(t, x) &= \{\beta(t) - D\vec{\Delta}\}\, V(t, x) + V(t, x)^* \{\beta(t)^* - D\overleftarrow{\Delta}\} \\ &\quad + \sigma^2 \delta(x) \end{aligned} \tag{3.11}$$

Here $D = \mathrm{diag}(D_i)$, and both $\vec{\Delta}$ and $\overleftarrow{\Delta}$ operate on $V(t, x)$.

(3.11) with (3.9), (3.10) under a boundary condition $V(t, x) = V(t, x + 2L)$, $\partial_x V(t, x) = \partial_x V(t, x + 2L)$ forms a closed set of equations in the EL.

4. EL FOR GINZBURG–LANDAU EQUATION

4.1. Basic Equations

Let us consider a case $b(z) = \mu z - \nu z^3$ ($\nu > 0$) in (2.1):

$$\partial_t X(t, x) = \mu X(t, x) - \nu X(t, x)^3 + D\, \partial_x^2 X(t, x) + \sigma f(t, x) \tag{4.1}$$

For simplicity we make the following scale transformation:

$$\begin{aligned} X &= (\mu_0/\nu)^{1/2}\, \bar{X} \\ \mu &= \mu_0 \bar{\mu} \\ t &= \bar{t}/\mu_0 \\ x &= (D/\mu_0)^{1/2}\, \bar{x} \end{aligned} \tag{4.2}$$

where $\mu_0 = (\nu^2\sigma^4/D)^{1/3}$. We then have an equation for $\bar{X}(\bar{t}, \bar{x})$ which makes the same form as (4.1) with $\mu = \bar{\mu}$, $\nu = \sigma = D = 1$. In the above derivation we have used the fact that, for all positive constants λ_1, λ_2, $f(\lambda_1 t, \lambda_2 x) \sim (\lambda_1\lambda_2)^{-1/2} f(t, x)$ where $\sim$ indicates the equality in the sense of distribution. Hereafter we assume $D = \nu = \sigma = 1$ unless otherwise stated.

Functions $\alpha(t)$, $\beta(t)$ defined by (2.7), (2.8) are now given by

$$\alpha(t) = \mu m(t) - 3m(t)\, V(t, 0) - m(t)^3 \tag{4.3}$$

$$\beta(t) = \mu - 3m(t)^2 - 3V(t, 0) \tag{4.4}$$

In deriving (4.3), (4.4) we have used the relations $\langle \xi(t, x)^{2n+1} \rangle = 0$ and $\langle \xi(t, x)^4 \rangle = 3\langle \xi(t, x)^2 \rangle^2 = 3V(t, 0)^2$, which follow from the fact that $\xi(t, x)$ is a Gaussian with mean zero.

Equations (2.9), (2.17) with (4.3), (4.4) are going to be analyzed.

4.2. Stationary Solutions

Stationary solutions $(m, V(x))$ satisfy

$$\mu m - 3mV(0) - m^3 = 0 \tag{4.5}$$

$$2\beta V(x) + 2\partial_x^2 V(x) + \delta(x) = 0 \tag{4.6}$$

where

$$\beta = \mu - 3m^2 - 3V(0) \tag{4.7}$$

Equation (4.6) is solved as

$$V(x) = \tfrac{1}{4}(-\beta)^{-1/2} \exp[-(-\beta)^{1/2}\, |x|] \tag{4.8}$$

β needs to be negative to guarantee $\lim_{|x| \to \infty} V(x) = 0$. From (4.5) it follows that $m = 0$ or $m^2 = \mu - 3V(0)$.

Suppose $m = 0$. Then $\beta = \mu - 3V(0) < 0$, i.e., $V(0) > \mu/3$. From (4.7), (4.8) we have an algebraic equation for $V(0)$:

$$48V(0)^3 - 16\mu V(0)^2 - 1 = 0 \tag{4.9}$$

which has a unique solution in $(\mu/3, \infty)$, say, V_A, for any real μ.

Suppose $m^2 = \mu - 3V(0)$. Then $\beta = -2(\mu - 3V(0)) < 0$, i.e., $0 < V(0) < \mu/3$. From (4.7), (4.8) we have

$$96V(0)^3 - 32\mu V(0)^2 + 1 = 0 \tag{4.10}$$

which has no solution in $(0, \mu/3)$ if $\mu < \mu_c \equiv 3(9/2)^{1/3}/4 = 1.238...$, and two solutions in $(0, \mu/3)$, say, V_B, V_C ($V_B \leqslant V_C$) if $\mu \geqslant \mu_c$. Using a pair of values $(m, V(0))$ to represent a stationary state itself, we have a unique stationary state $A = (0, V_A)$ if $\mu < \mu_c$, and five stationary states A, $B_\pm = (\pm m_B, V_B)$, $C_\pm = (\pm m_C, V_C)$ if $\mu \geqslant \mu_c$, where $m_i = (\mu - 3V_i)^{1/2}$, $(i = B, C)$. As will be seen in the Appendix, the states A and $B_\pm$ are stable under small fluctuations while $C_\pm$ are unstable.

The stationary distribution P_{st} of (4.1) is formally written in a path integral form[14]

$$P_{\mathrm{st}}(X(x) = X_0(x), x \in (-\infty, \infty))$$
$$= \mathcal{N} \exp \left\{ -2 \int_{-\infty}^{\infty} dx [U(X_0(x)) + \tfrac{1}{2}(dX_0(x)/dx)^2] \right\} \tag{4.11}$$

where

$$U(z) = -\int_0^z b(z)\, dz = -\tfrac{1}{2}\mu z^2 + \tfrac{1}{4}z^4 \tag{4.12}$$

and $\mathcal{N}$ is a normalization constant. Since the right-hand side of (4.11) is invariant under $X_0 \to -X_0$, stationary mean value of X is zero.

In the EL scheme, on the other hand, two locally stable states $B_\pm$ appear in addition to the stable state A when $\mu \geqslant \mu_c$. We recall that appearance of multiple steady states are also observed in the EL for stochastic ordinary differential equations.[2] The states $B_\pm$ should be called quasistationary since they correspond to states fluctuating around $\pm\mu^{1/2}$, the bottoms of the potential U (4.12); For μ sufficiently large, $\eta(t, x) = X(t, x) \pm \mu^{1/2}$ asymptotically satisfies

$$\partial_t \eta(t, x) = -2\mu\eta(t, x) + \partial_x^2 \eta(t, x) + f(t, x) \tag{4.13}$$

Correlation function $V_\eta(t, x) := \langle \eta(t, x)\, \eta(t, 0) \rangle$ follows

$$\partial_t V_\eta(t, x) = -4\mu V_\eta(t, x) + 2\partial_x^2 V_\eta(t, x) + \delta(x) \tag{4.14}$$

which has a stationary solution

$$V_\eta(x) = \tfrac{1}{4}(2\mu)^{-1/2} \exp[-(2\mu)^{1/2} |x|] \tag{4.15}$$

The stationary solution $(\mu^{1/2}, V_\eta(x))$ [resp. $(-\mu^{1/2}, V_\eta(x))$] asymptotically agrees with B_+ (resp. B_-); as $\mu \to \infty$, (4.10) says $V_B \sim \frac{1}{4}(2\mu)^{-1/2}$, so that $\beta = -2(\mu - 3V_B)$ in (4.8) $\sim -2\mu$.

Scalapino *et al.*[10] carried out numerical calculation by using the functional integral representation (4.11). In Fig. 1, stationary value $\langle X(0)^2 \rangle_{\mathrm{st}}$ obtained by the EL is compared with the one by Scalapino *et al.* Note that, in the EL,

$$\langle X(0)^2 \rangle_{\mathrm{st}} = m^2 + V(0) = \begin{cases} V_A & \text{in the state } A \\ \mu - 2V_B & \text{in the states } B_\pm \end{cases} \tag{4.16}$$

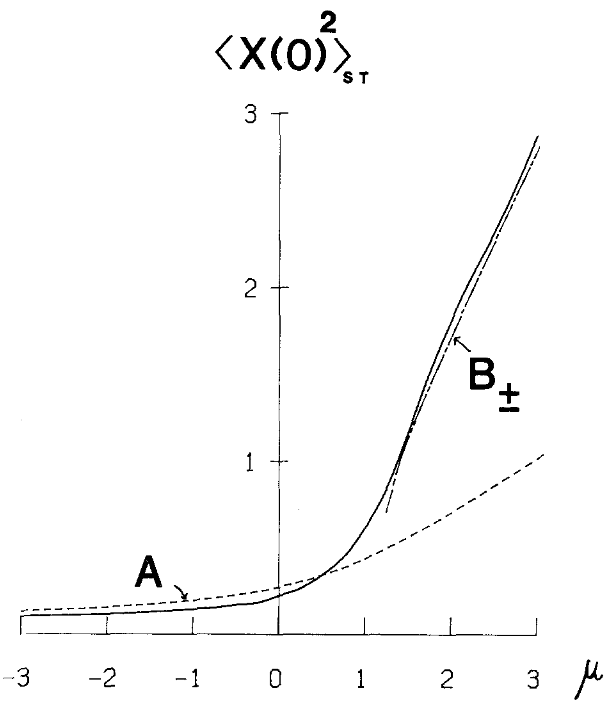

Fig. 1. Stationary value of second moment $\langle X(0)^2 \rangle_{\mathrm{st}}$ as a function of μ. Solid curve: exact result by Scalapino *et al.* Dotted curve: EL (states $A, B_\pm$).

Our result agrees with theirs for $\mu < 1$. After the appearance of multiple Steady states ($\mu > \mu_c$), the states $B_\pm$ give almost correct value; for μ large enough transition between two bottoms of the potential U seldom occurs, so that $\langle X(0)^2 \rangle_{\rm st}$ is obtained correctly if we consider U has only one well.

Stationary correlation functions are approximately expressed as[14]

$$I_1 := \langle X(x)\, X(0) \rangle_{\rm st} = \langle X(0)^2 \rangle_{\rm st} \exp(-|x|/L_1) \tag{4.17}$$

$$\begin{aligned} I_2 &:= \langle X(x)^2\, X(0)^2 \rangle_{\rm st} - \langle X(x)^2 \rangle_{\rm st} \langle X(0)^2 \rangle_{\rm st} \\ &= \{\langle X(0)^4 \rangle_{\rm st} - \langle X(0)^2 \rangle_{\rm st}^2\} \exp(-|x|/L_2) \end{aligned} \tag{4.18}$$

by using two correlation lengths L_1, L_2. In the EL,

$$I_1 = m^2 + V(x) \tag{4.19}$$

$$\begin{aligned} I_2 &= 4m^2 V(x) + \langle \xi(x)^2\, \xi(0)^2 \rangle - \langle \xi(x)^2 \rangle \langle \xi(0)^2 \rangle \\ &= 4m^2 V(x) + V(0)^2 + 2V(x)^2 - V(0)^2 \\ &= 4m^2 V(x) + 2V(x)^2 \end{aligned} \tag{4.20}$$

where $V(x)$ is given by (4.8). Hence we obtain

$$L_1^{-1} = \begin{cases} (\mu - 3V_A)^{1/2} & \text{for the state } A \\ 0 & \text{for the states } B_\pm \end{cases} \tag{4.21}$$

and

$$L_2^{-1} = \begin{cases} 2(\mu - 3V_A)^{1/2} & \text{for the state } A \\ \text{not the form (4.18)} & \text{for the states } B_\pm \end{cases} \tag{4.22}$$

Comparison between the result by the EL and the one by Scalapino *et al.* is given in Fig. 2. Agreement is good for both L_1 and L_2 if $\mu \lesssim 1$. For $\mu > \mu_c$ the correlation length L_1 in the state $B_\pm$ is almost correct, but the EL fails to give correct L_2 which strongly reflects the non-Gaussian feature.

4.3. Time Evolution

First we make a remark on the time evolution of the structure function defined by

$$\Phi(t, k) := \int dx\, e^{ikx} V(t, x) \tag{4.23}$$

By (2.16) we have

$$\Phi(t, k) = H(t)^2 \left\{ \int_0^t ds\, H(s)^{-2}\, e^{2k^2(s-t)} + \Phi(0, k)\, e^{-2k^2 t} \right\} \tag{4.24}$$

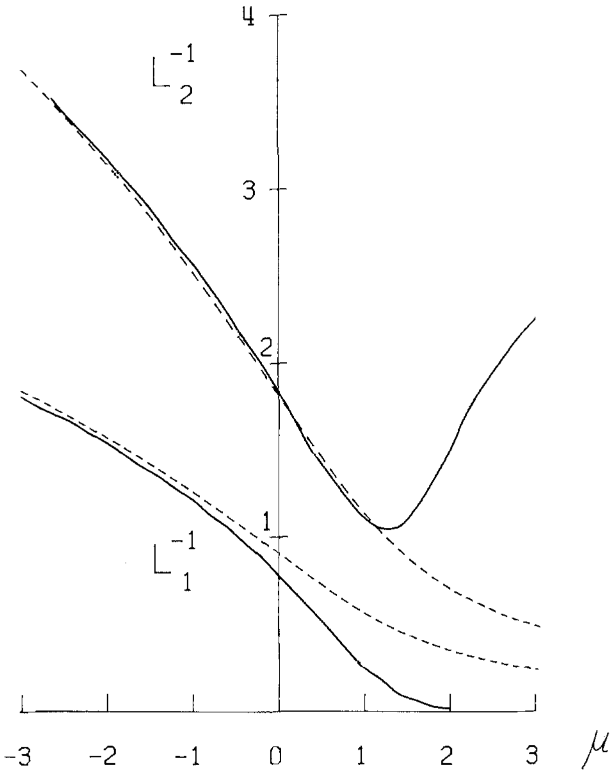

Fig. 2. Correlation length L_1 and L_2 as a function of μ. Solid curve: exact result by Scalapino *et al.* Dotted curve: EL (state A).

where $H(t)$ is given by (2.12). We remove the effect of initial distribution, i.e., $\Phi(0, k) \equiv 0$. Then (4.24) means that $\Phi(t, k)$ is a monotonically decreasing function of k, having a peak at $k = 0$. Half-width at half-maximum decays as $t^{-1/2}$ for small t since

$$\begin{aligned}
&\Phi(t, a^{1/2}t^{-1/2})\, \Phi(t, 0)^{-1} \\
&\quad = e^{-a} \int_0^1 ds\, e^{as} H(ts)^{-2} \left[t^{-1} \int_0^t ds\, H(s)^{-2} \right]^{-1} \to \tfrac{1}{2} \qquad \text{as} \quad t \to 0
\end{aligned}$$

where a is a positive root of an equation $1 - \exp(-a) = \frac{1}{2}a$. This fact has been pointed out by Tomita and Murakami,[8,9] who adopted an approximation different from the EL.

Let us next see how the diffusion term $\partial_x^2 X$ in (4.1) effects on the time evolution. Consider a set of equations in the EL[2-4]

$$\begin{aligned} \dot{m}_s(t) &= (\mu - 3V_s(t))\, m_s(t) - m_s(t)^3 \\ \dot{V}_s(t) &= 2\{\mu - 3m_s(t)^2 - 3V_s(t)\}\, V_s(t) + 1 \end{aligned} \tag{4.25}$$

for a stochastic differential equation

$$\partial_t X(t) = \mu X(t) - X(t)^3 + f(t) \tag{4.26}$$

Here m_s and V_s represent approximate mean and variance in the EL, and f is the Gaussian white noise.

Equations (2.9) and (2.17) with (4.3), (4.4), and (4.25) are numerically integrated when the system is initially set in a neighborhood of unstable point $x = 0$ of a dynamical system

$$\dot{x}(t) = \mu x(t) - x(t)^3 \tag{4.27}$$

i.e., initial conditions are $V(0, x) \equiv 0$, $m(0) \simeq 0$ for the model (4.1) and $V_s(0) = 0$, $m_s(0) \simeq 0$ for the model (4.26). Results given in Figs. 3 and 4 show that the diffusion term $\partial_x^2 X$ accelerates relaxation considerably when $m(0)$ and $m_s(0)$ are small.

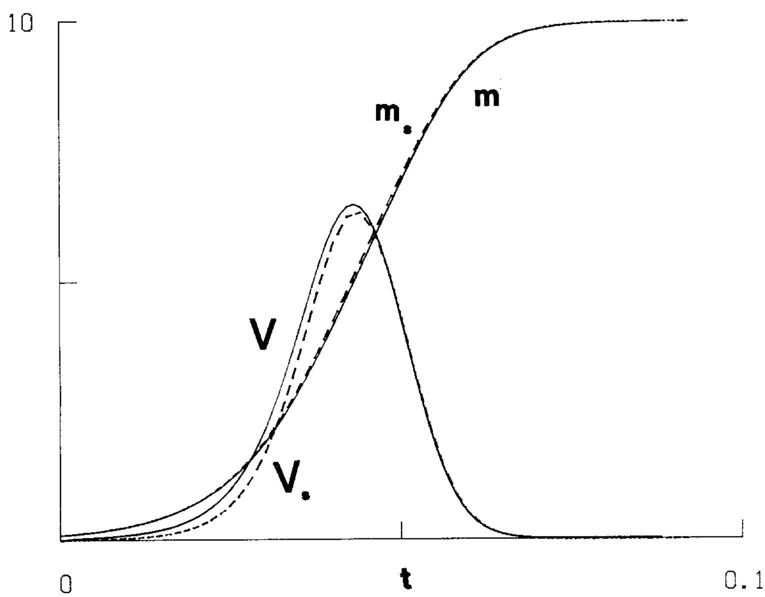

Fig. 3. Time development of mean and variance in the EL ($\mu = 10$). Solid curve: model (4.1). Dotted curve: model (4.25). Initial conditions are: $m(0) = 0.1$, $V(0, x) \equiv 0$ and $m_s(0) = 0.1$, $V_s(0) = 0$.

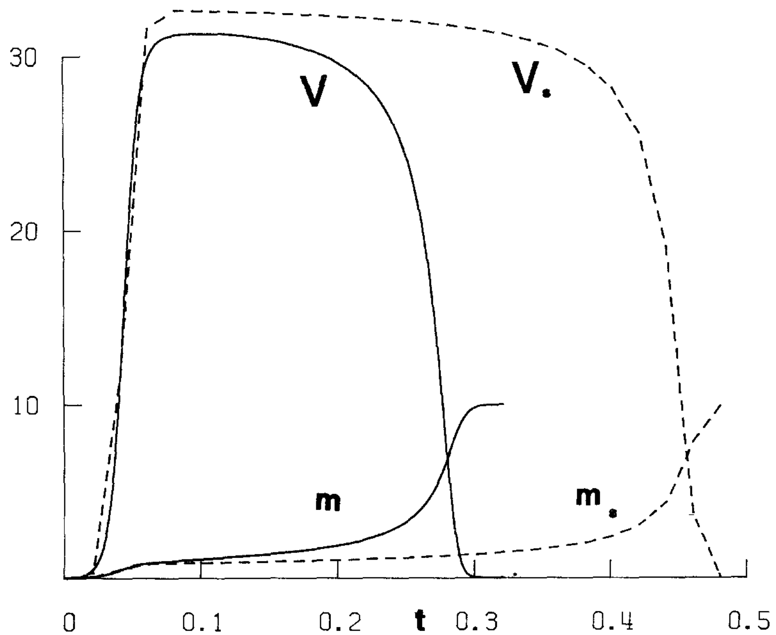

Fig. 4. Same as in Fig. 4 but $m(0) = 0.01$, $V(0, x) \equiv 0$ and $m_s(0) = 0.01$, $V_s(0) = 0$.

5. DISCUSSION

The field X following (1.4) will not have a property of spatial uniformity in the sense of ensemble average if it initially lacks. X may lose the property as time elapses even if it initially has. A better approximate Gaussian field $\tilde{Y}$ which describes spatial nonuniformity of X is sought as follows. For simplicity we consider $n = d = 1$. We suppose $\tilde{Y}$ follows an SPDE

$$\partial_t \tilde{Y}(t, x) = Z(t, x) + D\, \partial_x^2 \tilde{Y}(t, x) + \sigma f(t, x)$$

instead of (2.3). Here

$$Z(t, x) = \alpha(t, x) + \beta(t, x)[Y(t, x) - m(t, x)] + \gamma(t, x)\, \partial_x [Y(t, x) - m(t, x)]$$

Functions $\alpha(t, x)$, $\beta(t, x)$, $\gamma(t, x)$, $m(t, x)$ will be determined by minimizing

$$\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \langle [b(Y(t, x)) - Z(t, x)]^2 \rangle \, dx$$

Further analysis of such a complicated procedure is left to future works.

As seen from (4.11), the Ginzburg–Landau model has a stationary solution with the property of spatial uniformity in the sense of ensemble average. It remains unknown, however, that the model keeps the property as time evolves which the analysis of Section 4.3 implicitly assumed.

6. APPENDIX: STABILITY OF STATIONARY SOLUTIONS

By $(M, V(x))$ we denote one of the stationary states A, $B_\pm$, or $C_\pm$, i.e., it satisfies (4.5), (4.6) with $m^2 = M$. Linearizing evolution equations (2.9) and (2.17) with (4.3) and (4.4) around $(M, V(x))$, we have

$$\delta \dot{M}(t) = (2\mu - 6V(0) - 4M)\,\delta M(t) - 6M\,\delta V(t, 0) \tag{6.1}$$

$$\partial_t\,\delta V(t, x) = -6V(x)\{\delta M(t) + \delta V(t, 0)\} + 2(\beta + \partial_x^2)\,\delta V(t, x) \tag{6.2}$$

where β is given by (4.7). Let $\delta\hat{M}$, $\delta\hat{V}$ denote the Laplace transforms of δM, δV; $\delta\hat{M}(\lambda) := \int_0^\infty e^{-\lambda t}\,\delta M(t)\,dt$, $\delta\hat{V}(\lambda, x) := \int_0^\infty e^{-\lambda t}\,\delta V(t, x)\,dt$. Let $\delta\Phi$, $\delta\hat{\Phi}$ be the Fourier transforms of δV, $\delta\hat{V}$; $\delta\Phi(t, k) := \int_{-\infty}^{\infty} e^{ikx}\,\delta V(t, x)\,dx$, $\delta\hat{\Phi}(\lambda, k) := \int_{-\infty}^{\infty} e^{ikx}\,\delta\hat{V}(\lambda, x)\,dx$.

Laplace transformation of (6.1) in t yields

$$a_{11}\,\delta\hat{M}(\lambda) + a_{12}\,\delta\hat{V}(\lambda, 0) = -\delta M(0) \tag{6.3}$$

with

$$\begin{aligned} a_{11} &= 2\mu - 6V(0) - 4M - \lambda \\ a_{12} &= -6M \end{aligned} \tag{6.4}$$

Making the Laplace transformation in t and Fourier transformation in x, we have from (6.2)

$$\begin{aligned} \delta\hat{\Phi}(\lambda, k) = {} & -6\{\delta\hat{M}(\lambda) + \delta\hat{V}(\lambda, 0)\}\,G(\lambda, k)\,G(0, k) \\ & + \delta\Phi(0, k)\,G(\lambda, k) \end{aligned} \tag{6.5}$$

where

$$G(\lambda, k) := (2k^2 - 2\beta + \lambda)^{-1} \tag{6.6}$$

In deriving (6.5) we have used a relation $\int e^{ikx} V(x)\,dx = G(0, k)$, which follows from (4.8).

Put $\mathscr{K}(\lambda) = (\lambda - 2\beta)^{1/2}$ with $\mathrm{Re}\,\mathscr{K}(\lambda) > 0$. Then $\delta\hat{V}$ is calculated from (6.5) as

$$\begin{aligned} \delta\hat{V}(\lambda, x) = {} & 3 \cdot 2^{-1/2}\lambda^{-1}\{K(\lambda, x) - K(0, x)\}\{\delta\hat{M}(\lambda) + \delta\hat{V}(\lambda, 0)\} \\ & + \delta\Xi(\lambda, x) \end{aligned} \tag{6.7}$$

where

$$K(\lambda, x) = \exp[-2^{-1/2}\mathscr{K}(\lambda)\,|x|]/\mathscr{K}(\lambda) \tag{6.8}$$

and $\delta\Xi$ is the inverse Fourier transformation of the second term on the right-hand side of (6.5). In particular putting $x = 0$, we have

$$a_{21}\,\delta\hat{M}(\lambda) + a_{22}\,\delta\hat{V}(\lambda, 0) = \delta\Xi(\lambda, 0) \tag{6.9}$$

where

$$\begin{aligned} a_{21} &= 3 \cdot 2^{-1/2}\mathscr{K}(\lambda)^{-1}\,\mathscr{K}(0)^{-1}\{\mathscr{K}(\lambda) + \mathscr{K}(0)\}^{-1} \\ a_{22} &= 1 + a_{21} \end{aligned} \tag{6.10}$$

Let $A = (a_{ij})$ be a 2×2 matrix defined by (6.4) and (6.10). $\delta\hat{M}(\lambda)$, $\delta\hat{V}(\lambda, 0)$ are solved by (6.3) and (6.9), and $\delta\hat{V}(\lambda, x)$ is given by (6.7).

Note that $\det A = 0$ is a quartic equation of $\mathscr{K}(\lambda)$. Let z_j $(j = 1,..., 4)$ its solutions. Put $z_5 = (-2\beta)^{1/2}$, $z(s) = si(s;\text{real})$. $\delta V(\lambda, x)$ and $\delta M(t)$ are given by a linear combination of the inverse Laplace transformation of $g_i(\lambda) = (\mathscr{K}(\lambda) - z_i)^{-1}$ $(i = 1,..., 5)$, $g_s(\lambda) = (\mathscr{K}(\lambda) - z(s))^{-1}$, $g_i(\lambda)\exp[-2^{1/2}\mathscr{K}(\lambda)\,|x|]$ and $g_s(\lambda)\exp[-2^{1/2}\mathscr{K}(\lambda)\,|x|]$.

Let us calculate, for example, $\int_{c-i\infty}^{c+i\infty} g_j(\lambda)\,e^{\lambda t}\,d\lambda$ $(j = 1,..., 4)$. We consider an integration along a closed curve C on a λ plane with $-\pi < \arg(\lambda - 2\beta) \leqslant \pi$ shown in Fig. 5.

By residue theorem, we have

$$\int_C g_i(\lambda)\,e^{\lambda t}\,d\lambda = \begin{cases} 4\pi i z_i \exp[(2\beta + z_i^2)t] & \text{if} \quad \operatorname{Re}(z_i) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{6.11}$$

Contributions from C_1, C_2, C_6, C_7 tend to zero as $R \to \infty$. Total contribution from C_3, C_4, C_5 amounts at most $O(t^{-1}e^{2\beta t})$, which tends to zero as $t \to \infty$ since $\beta < 0$, so that terms effecting on the stability come from (6.11). Similar estimations are obtained from the inverse Laplace transformation of other functions. Stability criterion for δM, δV now becomes

$$\begin{cases} \text{unstable if } \operatorname{Re}(z_i) > 0 \text{ and } \operatorname{Re}(2\beta + z_i^2) > 0 & \text{for some} \quad i = 1,..., 4 \\ \text{stable} & \text{otherwise} \end{cases} \tag{6.12}$$

Put $w = w(\lambda) = (-2\beta)^{-1/2}\mathscr{K}(\lambda)$.

State A $[M = 0,\ \beta = \mu - 3V(0)$ where $V(0)$ is given by (4.9)]. Characteristic equation $\det A = 0$ reads

$$(\lambda - 2\beta)(w^2 + w + q) = 0 \tag{6.13}$$

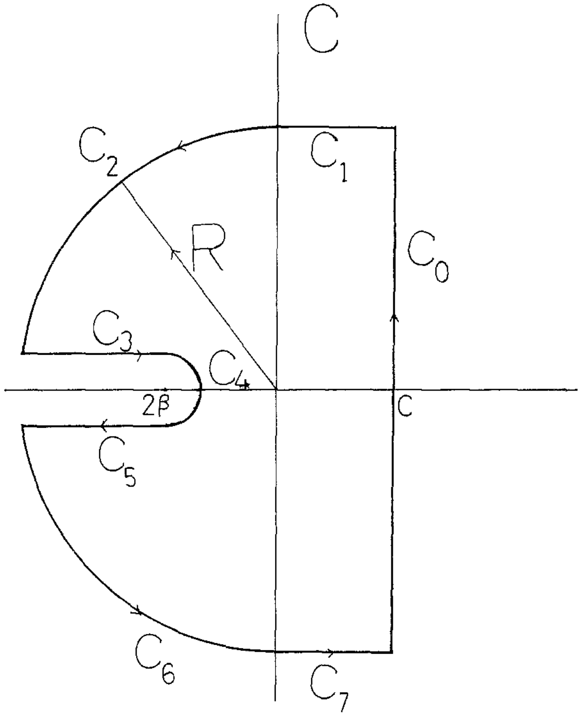

Fig. 5. Contour C in λ plane.

where $q = 3 \cdot 4^{-1}(-\beta)^{-3/2}$. By (6.12) we conclude A is stable.

States $B_\pm C_\pm$ $[M = \mu - 3V(0), \beta = \mu - 3M - 3V(0)$ where $V(0)$ is given by (4.10)]. Characteristic equation is given by

$$h(w) := (2w^2 - 1)(w^2 + w + q) - 3q = 0 \tag{6.14}$$

where $q = 3 \cdot 4^{-1}(-\beta)^{-3/2}$. Note $V_B < 2\mu_c/9 < V_C$, so that $q < 1$ for the states $B_\pm$, and $q \geqslant 1$ for the $C_\pm$. Since $h(1) = 2 - 2q \leqslant 0$ for $q \geqslant 1$, $h(w) = 0$ has a real solution in $[1, \infty)$; hence the states $C_\pm$ are unstable.

When $0 < q < 1$, we have $h(1) > 0$, $h(w)\uparrow$ for $w > 1$, so that $h(w) = 0$ has no solution for $w \geqslant 1$. Suppose w is complex: $w = a + bi$ $(a > 0, b \neq 0)$. From (6.14) we get

$$\mathrm{Re}(w^2) = a^2 - b^2 = \{-4a^2 - (4q - 2)a + 1\}/(8a + 2) \tag{6.15}$$

It is easy to see that inequality $a^2 - b^2 > 1$ is not satisfied for $0 < q < 1$, so that $B_\pm$ are stable.

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